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On estimating Weibull modulus by moments and maximum likelihood methods

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Abstract The effect of true Weibull modulus and sample size on Weibull modulus estimated by moments and maximum likelihood methods was investigated. Results indicated that the value of true Weibull modulus had no effect on estimated modulus for the maximum likelihood method, and a strong effect for the moments method, especially when sample size was less than 30. In addition, the distribution of Weibull modulus estimated by both methods was investigated using the modified Anderson–Darling statistics for goodness of fit. It was found that the distribution was not normal, lognormal, 3-parameter Weibull, or 3-parameter log-Weibull for the maximum likelihood method, as reported in previous studies. For the moments method however, the distribution of normalized Weibull moduli was found to be lognormal for sample sizes of 40 and above. The other three distributions showed a significant level of lack-of-fit at all sample sizes.

Introduction

Weibull statistics is widely used to model the variability in the fracture properties of ceramics and metals. The probability, P, that a part will fracture at a given stress, σ, or below can be predicted as [1];

$$P = 1 - \exp\left[-\left(\frac{\sigma - \sigma_T}{\sigma_0}\right)^m\right] \quad (1)$$

where σ_T is the threshold value below which no failure is expected, σ_0 is the scale parameter, and m is the Weibull modulus, alternatively referred to as the shape parameter. Equation 1 is for the 3-parameter Weibull distribution. When σ_T is taken as zero, Eq. 1 reduces to a 2-parameter Weibull cumulative probability function. The 2-parameter case is addressed in this study.

The value of the estimated Weibull modulus has been used as a measure of reliability in a variety of applications, one of which is the filling system design of aluminum alloy castings. Green and Campbell [2] showed that the tensile strength of cast Al–Si alloys follow a Weibull distribution and that the filling system design has a strong effect on the Weibull modulus. According to Campbell [3], m is often between 1 and 10 for pressure die castings, and between 10 and 30 for many gravity-filled castings. For good quality aerospace castings, m is expected to be between 50 and 100.

There are several methods available in the literature to estimate the Weibull modulus: linear regression (least square), weighted least square, moments method, and maximum likelihood method. The last two methods are the subject of this study.

In 2-parameter Weibull distributions, the ratio of average to standard deviation (also known as the signal-to-noise ratio) is written as

$$\frac{\bar{\sigma}}{s_\sigma} = \frac{\Gamma\left(1 + \frac{1}{m}\right)}{\sqrt{\Gamma\left(1 + \frac{2}{m}\right) - \left(\Gamma\left(1 + \frac{1}{m}\right)\right)^2}} \quad (2)$$

where $\bar{\sigma}$ is the average fracture stress, s_σ is the standard deviation and Γ represents the gamma function. When m is plotted as a function of the signal-to-noise ratio following Eq. 2, Fig. 1 is obtained, which shows a linear relationship and indicates the strong dependence of the combined effect of average and scatter on the 2-parameter Weibull

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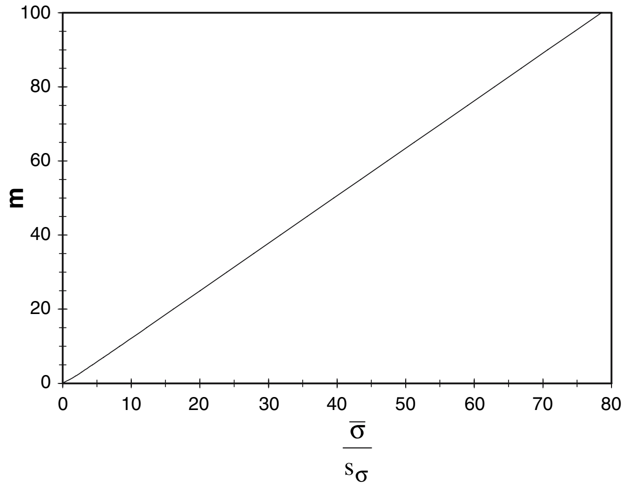

Fig. 1 The effect of the signal-to-noise ratio on the Weibull modulus

modulus. The Weibull modulus can be estimated from Eq. 2, which represents the moments method. For the maximum likelihood method, the equation to be solved is

$$\sum_{i=1}^{n} \ln(\sigma_i) - n\left(\frac{\sum_{i=1}^{n} \ln(\sigma_i)\sigma_i^m}{\sum_{i=1}^{n} \sigma_i^m}\right) + \frac{n}{m} = 0 \quad (3)$$

where n is the sample size. In both moments and maximum likelihood methods, the Weibull modulus is estimated by an iterative procedure (e.g., the Newton–Raphson method) until Eqs. 2 and 3 are solved.

Since the Weibull modulus is determined from a sample, the calculated modulus ($\hat{m}$) is only an estimate of the true modulus, m. Consequently, confidence limits often need to be attached to the estimated modulus, which necessitates that the distribution of $\hat{m}$ be known. Thoman et al. [4] showed that the distribution of $\hat{m}/m$ is independent from the value of m and is affected only by sample size for the maximum likelihood method. Khalili and Kromp [5] later confirmed this finding. They also used histograms of m estimated by the moments method for n = 30 at three m values (10, 20, and 100) and concluded qualitatively that the moments method also yields Weibull modulus estimates that are not affected by m. This finding for the moments method, however, contradicts the results of Trustrum and Jayatilaka [6]. For the maximum likelihood method, Yoon and Cho [7] found that m did affect the distribution of $\hat{m}/m$. In their study, they used m values between 3 and 30. Hence there seem to be conflicting results in the literature about these two methods.

The distribution of $\hat{m}$ has been investigated in several studies. Ritter et al. [8] ran Monte Carlo simulations only 100 times and concluded that the distribution of the estimated Weibull modulus is approximately normal. It has since been shown [4, 5, 9–12] that the distribution of $\hat{m}$ is positively skewed. Recently, Gong [9] revisited the distribution of $\hat{m}/m$ and found results similar to those of Thoman et al. Gong also stated that the distribution of $\ln(\hat{m}/m)$ obtained by the maximum likelihood method is normal, i.e., $\hat{m}/m$ follows a lognormal distribution. However, Gong did not report any goodness of fit test to support his findings. In a later publication, Gong and Wang [10] stated that $\hat{m}$ follows a lognormal distribution for linear regression and maximum likelihood methods. These authors used the χ^2 goodness-of-fit test for their evaluation. Barbero et al. [11] claimed that the distribution of $\hat{m}$ estimated by the maximum likelihood method is better expressed by a 3-parameter Weibull distribution. In a later publication [12], the same authors found that 3-parameter log-Weibull distribution provides a better fit to $\hat{m}/m$ estimated by the maximum likelihood method than lognormal and 3-parameter Weibull distribution. Barbero et al. did not provide the results of any hypothesis tests on the goodness of fit for the distributions that they suggested. Recently, Tiryakioğlu and Hudak [13] showed, by using the modified Anderson-Darling goodness of fit test, that the distribution of $\hat{m}/m$ estimated by the linear regression method is not normal, lognormal, 3-parameter Weibull, or 3-parameter log-Weibull. In addition, no study has addressed the distribution of Weibull moduli obtained by the moments method, to the knowledge of the author. These results indicate that there is a need for a systematic approach to characterizing the distribution of $\hat{m}/m$.

The goal of this study is to investigate the effect of m on the distribution of $\hat{m}/m$ systematically for the two methods for a wide range of m and n. In addition, the distributions suggested in the literature, i.e., normal, lognormal, 3-parameter Weibull, and 3-parameter log-Weibull, are tested in this study.

Experimental details

Monte-Carlo simulations were used to generate n data from a Weibull distribution with parameters $\sigma_0 = 1$ and 11 levels of m: 1, 2, 3, 5, 7, 10, 20, 30, 50, 75, and 100, covering the entire range of Weibull moduli expected in aluminum castings. Seven sample sizes were used in this study: 5, 10, 15, 20, 30, 40, and 50. For one observation, n random numbers between 0 and 1 were generated to obtain a set of σ values. The Newton–Raphson method was used to solve for m to satisfy Eqs. 2 and 3. For each sample size and m, the experiment was repeated 10,000 times. Estimated Weibull moduli were normalized by dividing them by m. Subsequently, average (M) and standard deviation (s_m) of normalized moduli were calculated.

Results and discussion

The effect of m on M and s_m

The effect of m and n on M and s_m for the moments method are shown in Figs. 2 and 3, respectively. In general, both M and s_m determined by the moments method are affected by the value of m. The magnitude of the effect on M is highest at low n and m. Only for $m \geq 20$ and $n \geq 15$, M and s_m for the moments method are unaffected by m. This helps explain why Khalili and Kromp did not find a difference in their histograms (n = 30 and m = 10, 20, and 100).

The effect of n and m on M and s_m for the maximum likelihood method are shown in Figs. 4 and 5, respectively. M is affected only by n and is independent of m. The same can be stated for s_m, although there seems to be a very slight effect at low values of m. These results agree with those of Thoman et al. but bring into doubt those of Yoon and Cho.

Figure 6 shows that the effect of n on the distribution of $\hat{m}/m$ when Eq. 2 is used. Note that the distribution becomes more normal with increasing n. The same observations were made for the histograms for the moments method

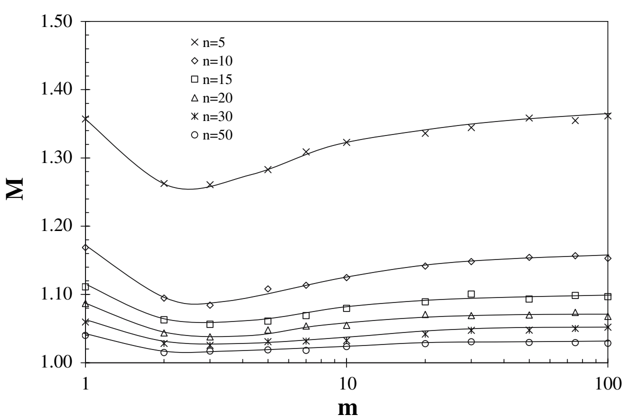

Fig. 2 The effect of m and n on M for the moments method

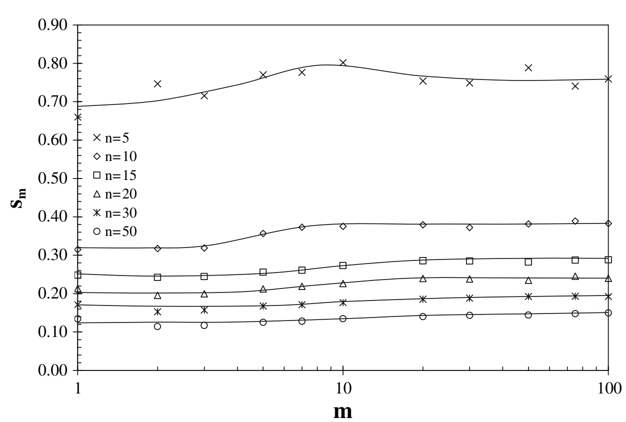

Fig. 3 The effect of m and n on s_m for the moments method

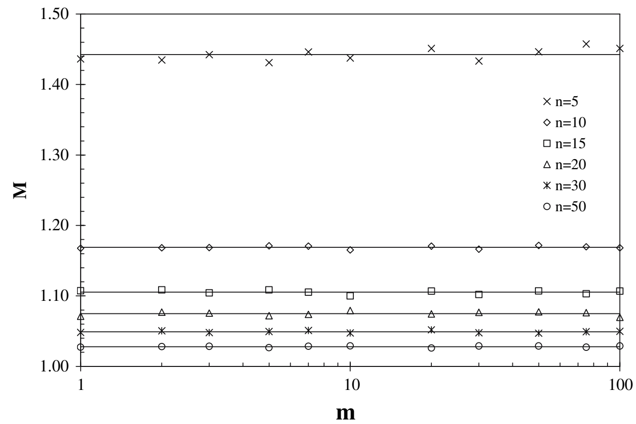

Fig. 4 The effect of m and n on M for the maximum likelihood method

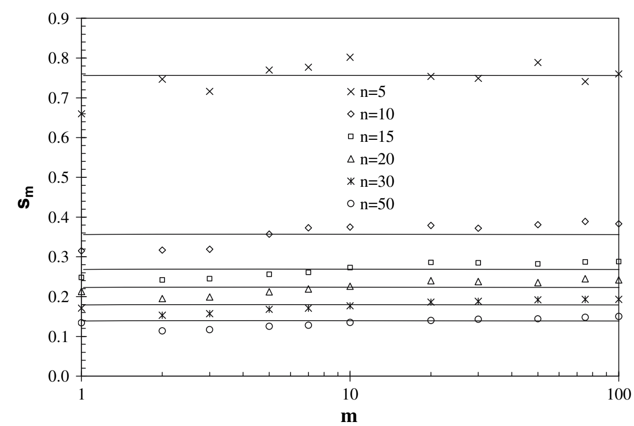

Fig. 5 The effect of m and n on s_m for the maximum likelihood method

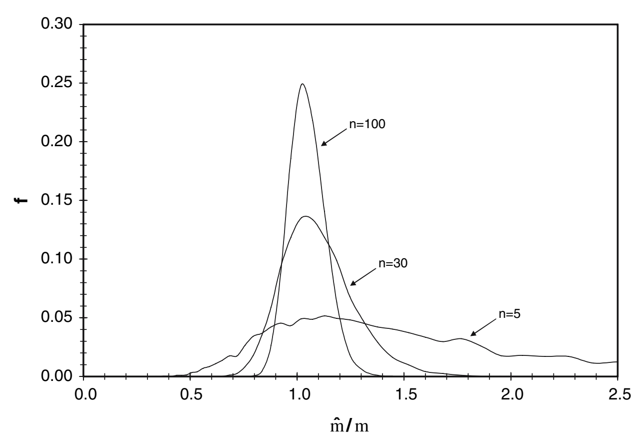

Fig. 6 Histograms of $\hat{m}/m$ for the maximum likelihood method at three sample sizes

The distribution of m

To determine whether m, estimated by the moments and maximum likelihood methods, follows the normal, log-normal, 3-parameter Weibull, or 3-parameter log-Weibull distributions, hypothesis tests were conducted using the modified Anderson–Darling (A^2) goodness-of-fit test statistic [14, 15]:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)\ln P_i + (2n-1-2i)\ln(1-P_i)\right] \tag{4}$$

The Anderson–Darling test was selected because of its sensitivity to the tails of the distribution. The less the value of A^2, the higher the confidence that data follow the distribution being tested. Normalized moduli obtained with both methods at $m = 30$ were analyzed. The results are presented in Tables 1 and 2 for the moments and maximum likelihood methods, respectively, in which A^2 test statistics and corresponding p-values are reported for each fit. The hypothesis that the dataset follows the tested distribution can be rejected only when p-value is less than a specified value for Type I error (α), which is typically prescribed as 0.05.

For the moments method, the normal, 3-parameter Weibull and 3-parameter log-Weibull distributions were rejected at every sample size. However, the hypothesis that the distribution of $\hat{m}/m$ is lognormal could not be rejected for $n = 40$ and 50. The trend of A^2 for the lognormal distribution suggests that P-value exceeds the critical point of 0.05 at a sample size slightly less than 40. However this sample size was not investigated in this study.

Results in Table 2 show that the distribution of $\hat{m}/m$ obtained by the maximum likelihood method is neither of the four distributions for sample sizes tested in this study. Hence the results of Gong [9], Gong and Wang [10], and Barbero et al. [11, 12] are in question.

Probability plots for the four distributions when $n = 50$ are presented in Figs. 7–10. Note that fits to the data obtained by the moments method show a significant level of lack-of-fit, except in Fig. 8 where the lognormal fit follows the trend of the data closely. For the maximum likelihood method, however, data show deviation from the fit at both tails for all four distributions, similar to results of Tiryakioğlu and Hudak [13] using the linear regression method. Therefore, of the three methods of estimating the Weibull modulus, only the moments method yields a formal distribution (for $n \geq 40$).

Table 1 A^2 values for the four distributions fitted to $\hat{m}/m$ data by the moments method

n	Normal		Lognormal		3-p Weibull		3-p Log-Weibull	
	A^2	p	A^2	p	A^2	p	A^2	p
5	405.24	<0.005	16.38	<0.005	143.06	<0.005	34.02	<0.005
10	131.56	<0.005	2.89	<0.005	74.36	<0.005	20.42	<0.005
15	83.66	<0.005	1.65	<0.005	45.7	<0.005	11.67	<0.005
20	61.09	<0.005	1.13	0.006	50.34	<0.005	16.02	<0.005
30	35.84	<0.005	0.97	0.015	42.88	<0.005	15.68	<0.005
40	27.37	<0.005	0.61	0.113*	37.12	<0.005	14.91	<0.005
50	18.16	<0.005	0.19	0.901*	23.63	<0.005	8.43	<0.005

All values except those with an asterisk indicate a significant degree of lack of fit

Table 2 A^2 values for the four distributions fitted to $\hat{m}/m$ data by the maximum likelihood method

n	Normal		Lognormal		3-p Weibull		3-p Log-Weibull	
	A^2	p	A^2	p	A^2	p	A^2	P
5	435.49	<0.005	32.68	<0.005	133.82	<0.005	28.46	<0.005
10	166.91	<0.005	14.81	<0.005	94.41	<0.005	32.11	<0.005
15	114.31	<0.005	13.91	<0.005	60.35	<0.005	20.32	<0.005
20	79.58	<0.005	8.91	<0.005	45.67	<0.005	15.29	<0.005
30	40.04	<0.005	3.36	<0.005	38.34	<0.005	15.85	<0.005
40	31.95	<0.005	2.95	<0.005	41.64	<0.005	19.49	<0.005
50	24.23	<0.005	1.96	<0.005	30.11	<0.005	13.03	<0.005

All values indicate a significant degree of lack of fit

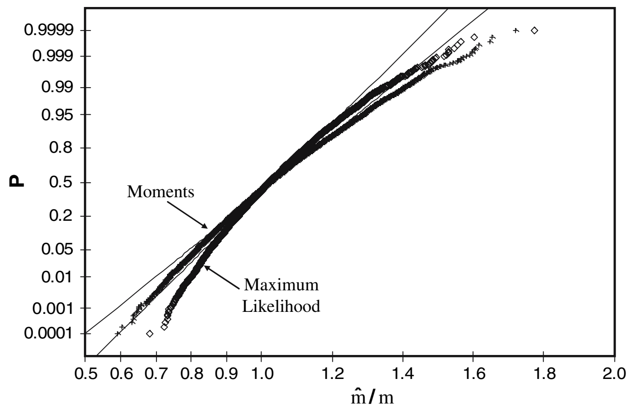

Fig. 7 Normal probability plot of $\hat{m}/m$ results for n = 50 and m = 30

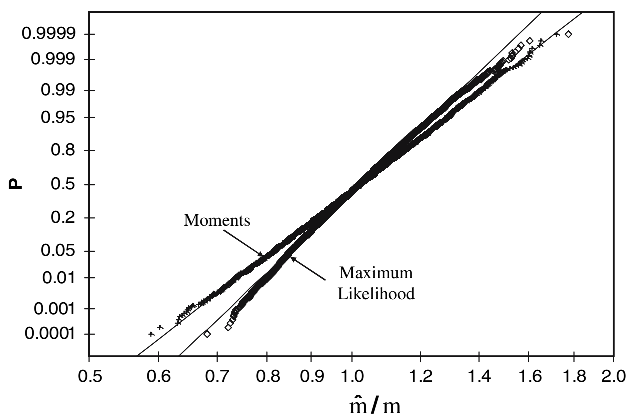

Fig. 8 Lognormal probability plot of $\hat{m}/m$ results for n = 50 and m = 30

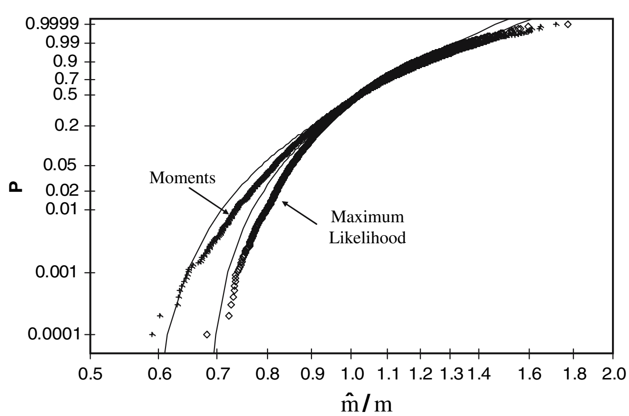

Fig. 9 3-Parameter Weibull probability plot of $\hat{m}/m$ results for n = 50 and m = 30

To the author's knowledge, this is the first time that a known distribution could be fitted to normalized Weibull modulus data. Therefore critical points of the lognormal distribution can be calculated directly to establish

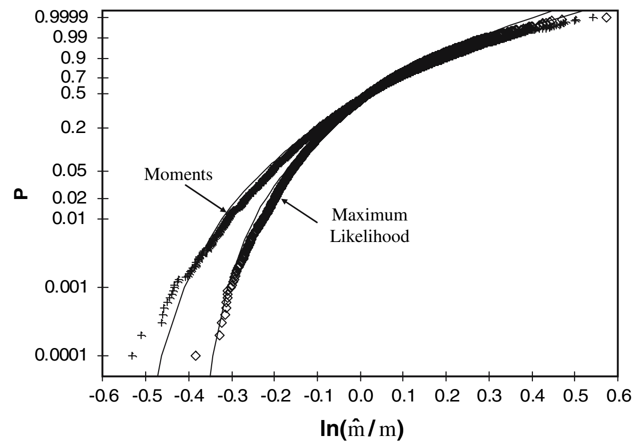

Fig. 10 3-Parameter log-Weibull pobability plot of $\hat{m}/m$ results for n = 50 and m = 30

confidence limits on Weibull moduli estimated by the moments method.

Conclusions

- The distribution of normalized Weibull moduli determined by the moments method is affected by not only the sample size but also m. This method should be used only when $n \geq 15$ and Weibull modulus is estimated to be larger than 20.
- For the maximum likelihood method, distributions of normalized moduli are affected only by the sample size.
- The distribution of normalized Weibull modulus estimated by the maximum likelihood method is not normal, lognormal, 3-parameter Weibull, or 3-parameter log-Weibull, as previously suggested in the literature.
- The distribution of normalized modulus estimated by the moments method is not normal, 3-parameter Weibull, or 3-parameter log-Weibull for sample sizes ranging from 5 to 50.
- The moments method yields Weibull moduli that are distributed lognormally for sample sizes 40 and higher.

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